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THE BOUNDARY LAYER BETWEEN A PLASMA AND A MAGNETIC FIELD - I

Harold Grad

December 28, 1960

AEC Research and Development Report

NEW YORK UNIVERSITY

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Magneto-Fluid Dynamics Division
Institute of Mathematical Sciences
New York University

Physics and
Mathematics

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Abstract

The problem of a steady boundary layer or sheath between a plasma and a magnetic field is considered. A self-consistent transition layer is found which joins a uniform magnetic field at plus infinity with a collisionless field-free plasma region with arbitrary velocity distribution at minus infinity; i.e., a magnetic field profile is found such that the exact particle orbits in this field produce a current which gives rise to this field. An interesting feature of the solution is that, with any nonsingular velocity distribution at minus infinity, the magnetic field penetrates only a finite distance into the plasma, although the plasma extends to infinity, exponentially attenuated, into the magnetic field region. The scale of length is the Larmor radius. Electric fields arising from charge separation in the case of particles of different mass are ignored in this paper.

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THE BOUNDARY LAYER BETWEEN A PLASMA AND A MAGNETIC FIELD - I*

1. Introduction

An idealization which has proved to be extremely useful is that of an interface or current sheet separating a perfectly conducting fluid from a magnetic field. Equilibrium is characterized by equality of the magnetic pressure, $B^2/2\mu_o$, and the fluid pressure, p. Analysis of this type of equilibrium configuration can be accomplished by classical, powerful mathematical tools[1]; analysis of stability was at once the earliest and most completely developed[2]; and in the form of the Cusped Geometry, such a model offered the earliest example of an absolutely stable confined conducting fluid[3].

Nevertheless, there are many purposes for which this model is inadequate and some measure of the thickness of the transition layer or even its fine structure is required. This is necessary, for example, to compute particle losses from a Cusped

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[*] This work was done largely in 1955 in connection with the problem of estimating particle losses in cusped geometries.

[1] H. Grad, WASH-184, p. 144 (Jan. 1955); also H. Grad and H. Rubin, Proc. Second International Conf. on the Peaceful Uses of Atomic Energy, Vol. 31, p. 190 (1958) and J. Berkowitz et al. Ibid. p. 171.

[2] M. D. Kruskal and M. Schwartzschild. Proc. Roy. Soc. A223, 348 (1954);

H. Grad, WASH-289, p. 115 (June 1955) and TID-7503, p. 238 (Feb. 1956); J. Berkowitz et al., Proc. Second International Conf. on the Peaceful Uses of Atomic Energy, Vol. 31, p. 177 (1958).

[3] H. Grad, WASH-289, p. 115 (June 1955).

Geometry, to refine stability criteria, and to estimate electron cyclotron radiation.

For simplicity, we consider an essentially one-dimensional model in which all quantities depend on the single coordinate x (traversing the sheath), and the magnetic field is uni-directional, say in the direction z, $B_z(x)$. The existence of an unambiguous solution of this type requires several idealizations and specializations. First, to allow such a time-independent solution requires the neglect of dissipative processes (macroscopically -- resistivity; microscopically -- particle collisions). Next we observe that the problem, as posed, does not have a unique solution. In the macroscopic version, any pressure profile $p(x)$ and magnetic field profile $B_z(x)$ satisfying the simple condition $p + B_z^2/2\mu_o$ = constant defines an allowable equilibrium solution. Similarly, we shall see that there is a correspondingly very large variety of microscopic solutions depending roughly on how many particles are placed on each magnetic line.

This ambiguity can be removed by asking for the _thinnest_ transition zone compatible with certain given properties at infinity. More precisely, it is shown that there exists a unique solution corresponding to any given velocity distribution at minus infinity, provided that there are no "trapped" particles which do not have access to infinity. Such a solution can be interpreted as the microscopic solution which best represents the originally described macroscopically sharp interface. But more important than this purely formal property is the significance of the thinnest layer in yielding the smallest possible particle loss rate in a Cusped Geometry and also the lowest possible radiation loss rate in any contained plasma. It is therefore a theoretical ideal which one should try to approximate experimentally.

It is very easy to estimate the order of magnitude of the thickness of this layer. The idealized current sheet requires an infinite current density. But, for a given mean thermal speed C, the maximum current density that can exist in the plasma is approximately

$$J = neC.$$

Thus, in a layer of thickness δ, the maximum change in B is given by

$$B = \mu_o J\delta = \mu_o neC\delta.$$

Ignoring constants of order unity, we have

$$p = nkT = nmC^2$$

from which

$$\delta = \frac{B}{\mu_o neC} = \frac{mC}{eB}\frac{B^2}{\mu_o p} .$$

Consequently, a change in B from the value zero to a value giving equality of magnetic and gas pressure can occur only over a distance at least as large as the representative Larmor radius, mC/eB. This argument, which makes no direct reference to particle orbits, is consistent with the concept of a Larmor radius as the minimum distance required to turn back a particle incident on a magnetic field. Furthermore, we shall see that the statement of self-consistency, viz., that the computed particle orbits give rise to the precise current required to produce the field gradient, is exactly equivalent to a more refined version of a pressure balance.

2. Particle Orbits

We consider a magnetic field with a single component in the z-direction and represent the component unambiguously

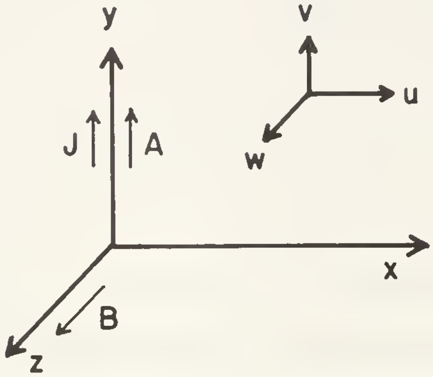


FIG. 1

by the scalar symbol B. B is assumed to vary in the x-direction only. Thus both the vector potential and the current can be taken in the y-direction, and we represent them by the scalar symbols A and J respectively; we have

$$(2.1)\quad \begin{cases} B = \dfrac{dA}{dx} \\[2em] J = \dfrac{1}{\mu_o}\dfrac{dB}{dx} = -\dfrac{1}{\mu_o}\dfrac{d^2A}{dx^2}\ . \end{cases}$$

The z-component of velocity of any particle is a constant; we are therefore free to consider the motion to be in the plane (x,y). The two dimensional Hamiltonian governing the motion of a particle of mass m and charge e (signed) is

$$H = \frac{1}{2m} [p_x^2 + (p_y - eA)^2] \tag{2.2}$$

where

$$\begin{cases} p_x = mu = m \dfrac{dx}{dt} \\ \\ p_y = mv + eA = m \dfrac{dy}{dt} + eA \end{cases} \tag{2.3}$$

Since the y-coordinate is ignorable, the problem can be treated as one-dimensional with the same Hamiltonian (2.2), but with p_y now interpreted as a given constant of the motion. As a one-dimensional system, the energy integral, H = constant, immediately gives a general solution explicit in terms of a quadrature involving the presumed given function $A(x)$ as well as the two constant values of H and p_y. However, almost all features of interest can be seen from the energy equation itself before quadrature; viz.,

$$\frac{1}{2} mu^2 + \Phi(x) = \epsilon \tag{2.4}$$

where

$$\Phi(x) = \frac{1}{2m} [\mu - eA(x)]^2. \tag{2.5}$$

We use the symbols ϵ and μ for the constant values taken by H and p_y respectively.

The motion is that of a particle under the influence of the potential Φ. For example, if the range of values taken by A is sufficiently great, the inequality $\Phi < \epsilon$ implies that the entire motion lies within the bounded interval on the x-axis determined by

$$(2.6) \qquad [\mu - eA(x)]^2 \leq 2m\epsilon .$$

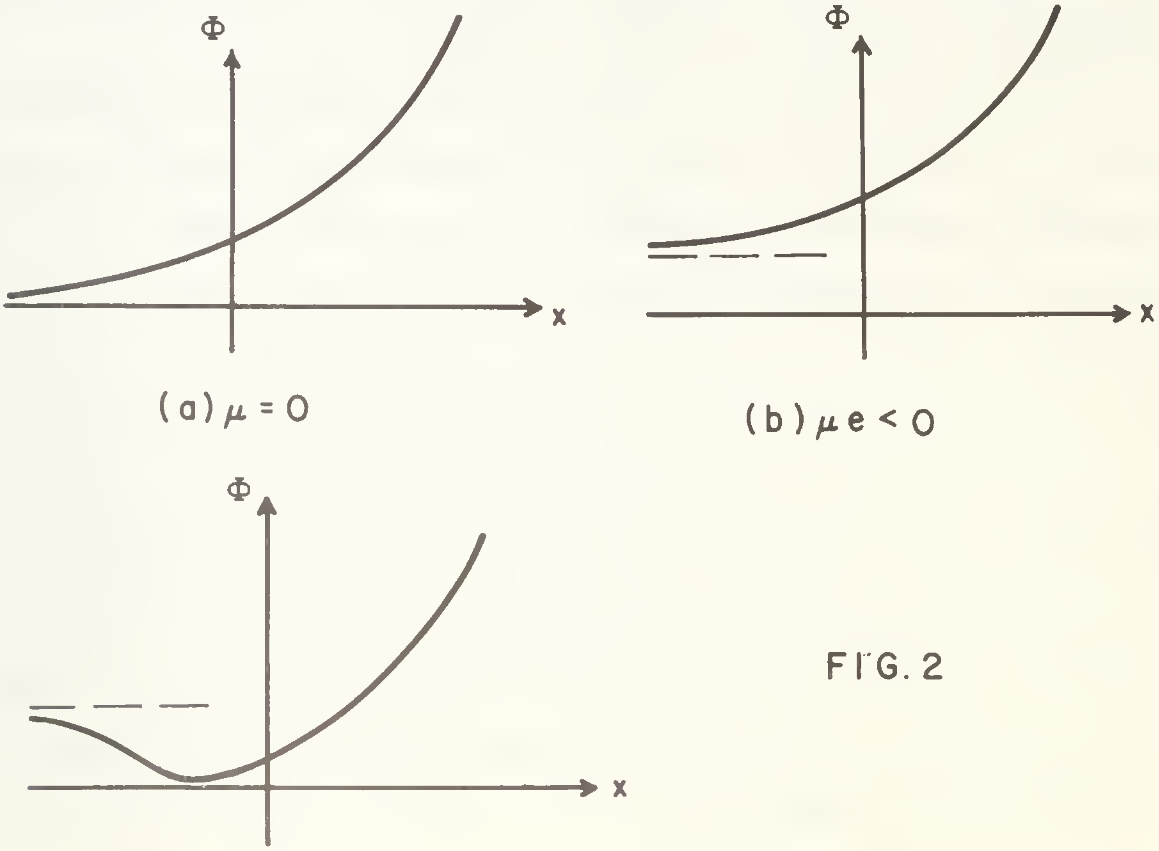
On the other hand, if A remains bounded as x recedes to plus or minus infinity, then for appropriate values of the constants ϵ and μ, infinity may be accessible to the particle in its motion. Such statements can be made precise if certain qualitative features of $A(x)$ are known.

To proceed further, we restrict our attention to special magnetic field profiles which are qualitatively similar to the expected solution of our problem. Specifically, $B(x)$ is positive and approaches the limit zero at minus infinity and the finite limit B_1 at plus infinity; in particular, $J = 0$ at both plus and minus infinity. We also expect B to be monotone, but this restriction is not required to carry out the analysis. If we further assume (to be verified afterwards) that $\int_{-\infty}^{x} B(x)dx$ converges at the lower limit, then $A(x)$ converges to a finite value at minus infinity, which value we can take to be zero. $A(x)$ increases monotonely and is unbounded at plus infinity. Concisely,

$$(2.7)\quad \begin{cases} A(-\infty) = 0, & A(+\infty) = +\infty, \quad A'(x) > 0 \\ B(-\infty) = 0, & B(+\infty) = B_1, \quad B(x) > 0 \\ J(-\infty) = 0, & J(+\infty) = 0 \end{cases}$$

It should be emphasized that these are properties which we hope the solution will have. Whether they are in the nature of boundary conditions that can be imposed at will, or whether they may be automatically satisfied or are even incompatible remains to be seen.

Adopting these restrictions on $A(x)$, we can now make much stronger statements about the possible orbits. For the potential $\Phi(x)$, we have the three possibilities shown in Figure 2 depending on whether μ has the same or opposite



(a) $\mu = 0$

(b) $\mu e < 0$

(c) $\mu e > 0$

FIG. 2

sign as e. In the two cases $\mu e \leq 0$, Φ is monotone. For any value of ϵ consistent with the existence of a real orbit, the particle "rolls up" the incline, turns around, and returns to minus infinity. The velocity, u, decreases monotonely (in time) through the value zero and ultimately approaches the negative of its initial value. The turning point, $u = 0$, is given by the unique solution of the equality (2.6). In the case $\mu e > 0$, a sufficiently small value of ϵ yields a periodic orbit in which the two extremes of x are given by the equality (2.6). On the other hand,a large value of ϵ yields an orbit similar to the previous ones, except that u is no longer monotone. We call the periodic orbits trapped and the ones accessible to infinity free.

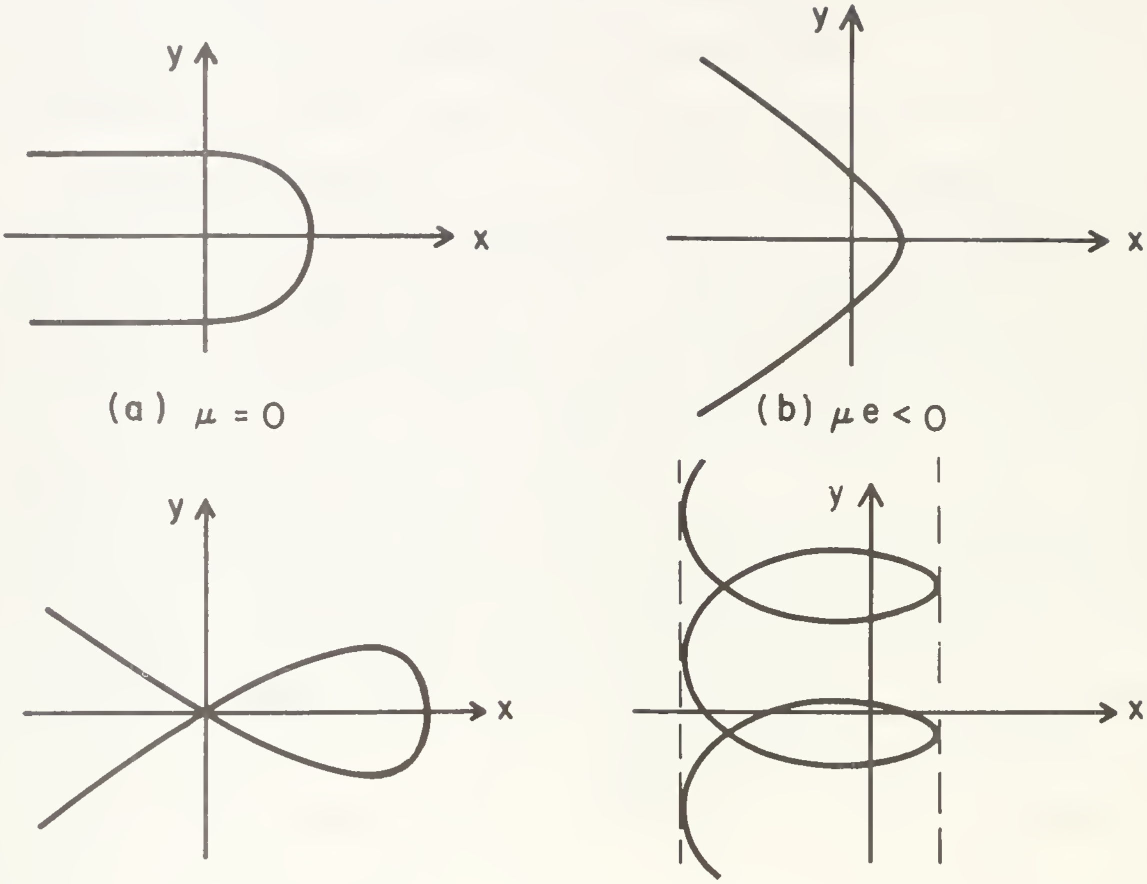
Quantitatively, the condition that an orbit be free is simply

$$(2.8) \qquad \epsilon \geq \mu^2/2m.$$

In the case $\mu e \leq 0$, there are no orbits which violate this inequality; in the case $\mu e > 0$, the trapped orbits are characterized by

$$(2.9) \qquad 0 \leq \epsilon < \mu^2/2m.$$

We shall find that the particle orbit properties just given are the only ones that enter into a full determination of the self-consistent solution. For completeness we show in Fig. 3 a representative set of orbits in the (x,y)-plane and, in Fig. 4, typical phase plane orbits; in the latter, μ is constant in each figure while ϵ is varied.



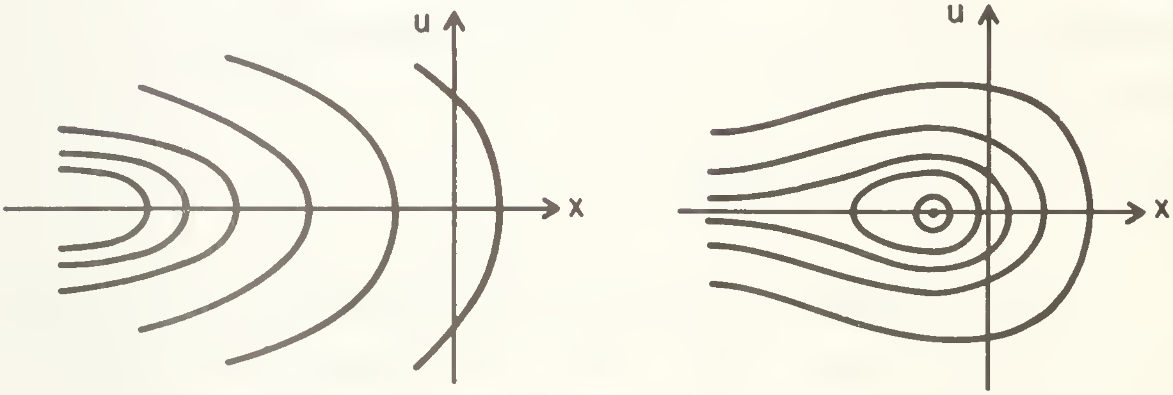
(a) $\mu = 0$

(b) $\mu e < 0$

(c) $\mu e > 0$, $\epsilon > \mu^2/2m$

(d) $\mu e > 0$, $\epsilon < \mu^2/2m$

FIG. 3



(a) $\mu e < 0$

(b) $\mu e > 0$

FIG. 4

3. General Self-Consistent Formulation

We are looking for a configuration or assemblage of charged particles which will be described by a time-independent distribution function $f(x,u,v)$. The dependence on the third component of velocity, w, has been omitted[4]. We normalize f to be a number density and note the macroscopic connections,

$$n(x) = \int f \, du \, dv \tag{3.1}$$

$$J(x) = e \int v f \, du \, dv \tag{3.2}$$

$$P_{xx}(x) = m \int u^2 f \, du \, dv. \tag{3.3}$$

The quantity P_{xx} is the indicated component of the stress tensor. If f is isotropic, it is exactly the pressure, p. It should be noted that, in virtue of symmetries that will appear, the macroscopic mean flow velocity in the x-direction is zero, $\int uf \, du \, dv = 0$.

The distribution function f satisfies Liouville's equation (sometimes called the collisionless Boltzmann equation or the Vlasov equation) which states merely that f is a constant along every curve in phase space which is the possible orbit of a particle. In this problem, it suffices to take (x,u,v) as the phase space instead of (x,y,u,v) which is, a priori, correct, since it is easily seen that there is a unique orbit which passes through each finite point (x,u,v).[5] This entire family of orbits is, in principle, known if A(x) is supposed to be given. A solution of Liouville's equation will then be specified by assigning values to f on each orbit.

---

[4] To be more realistic, one can consider f to be an even function of w; this condition for all x is compatible with the equations of motion. Thus there will be no resultant gas velocity or current in the z-direction, and all the results hold as given.

Now, we know that ϵ and μ are constant on an orbit. This suggests that we use them as parameters to specify the various orbits. The statement that f is constant on an orbit could then be interpreted as a statement that f is a function of ϵ and μ. This is approximately true, but requires elaboration. It is not necessarily true that a given pair of values (ϵ,μ) singles out a unique orbit. If A(x) is not monotone, there will be distinct sets of orbits with the same values of ϵ and μ; clearly, one may wish to assign different values to f on the different sets of orbits. It is easy to verify, however, that if A(x) is monotone, this assignment is unique. Geometrically, this is clear from Fig. 2 and the related discussion.

To effect the integration over f, (3.1)-(3.3), it is convenient to introduce ϵ and μ as integration variables instead of u and v. From the defining relations

$$(3.4) \qquad \begin{cases} \epsilon = \frac{1}{2} m(u^2 + v^2) \\ \mu = mv + eA \end{cases}$$

---

[5] In this context, the point u = v = 0, x = constant is to be interpreted as an orbit. These singular orbits can be ignored in the problem at hand since they do not contribute to the current.

in which A enters as a parameter, we see that the quantities n, J, and P_{xx} will be given as functions of A alone. The self-consistency relation, (2.1), takes the form of a second order differential equation, viz., d^2A/dx^2 is a specified function of A. There is an alternative procedure, viz., to use the exact macroscopic equation of momentum conservation,

$$P_{xx} + \frac{1}{2\mu_o} B^2 = \text{constant}. \tag{3.5}$$

This is a first order equation expressing $B = dA/dx$ as a function of A and is, or course, merely the first integral of the previous second order equation. However, it will be illuminating to carry through both procedures explicitly.

Before carrying out this plan, we must specify the distribution function, f. This is to a large extent arbitrary. For those values of ϵ and μ which correspond to free orbits extending to minus infinity, this specification amounts to a boundary condition on f at minus infinity. Thus a complete specification of f amounts to the imposition of a boundary condition plus the specification of the density of trapped particles for each appropriate ϵ and μ. The value of μ locates the potential minimum, Fig. 2c, and thus the approximate position of the trapped particle. The separation into free and trapped regions is shown in the (ϵ,μ)-plane in Fig. 5 and in the (u,v)-plane for a specific value of A in Fig. 6. Fig. 5 is drawn for $eA > 0$; the labels "trapped" and "no orbits" are to be interchanged if $eA < 0$. Similarly, in Fig. 6, the parabola opens downward if $eA < 0$.

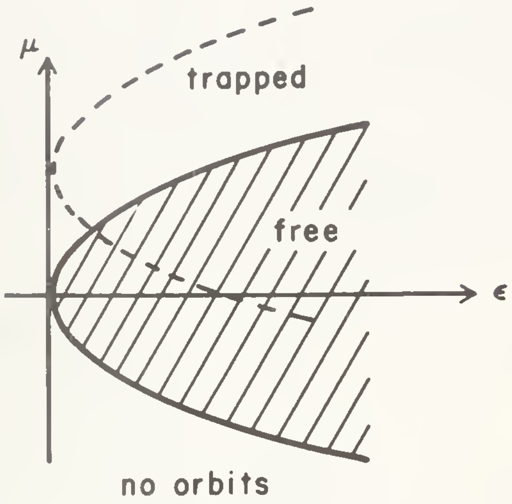


FIG.5

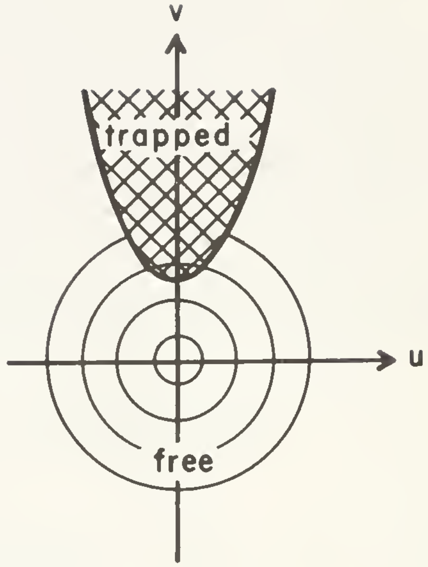


FIG. 6

We now return to the transformation (3.4) from the variables (u,v) to the variables (ϵ,μ). It is evident that each of the half planes $u > 0$ and $u < 0$ is mapped into the same domain of (ϵ,μ). Specifically, each half plane is mapped one-to-one into the parabola

$$(3.6) \qquad 2m\epsilon > (\mu - eA)^2 .$$

This multiple mapping merely introduces a factor two in the Jacobian; (the integrand in all the expressions (3.1)-(3.3) is even in u). We compute (for $u > 0$)

$$(3.7) \qquad \frac{\partial(\epsilon,\mu)}{\partial(u,v)} = \begin{vmatrix} mu & mv \\ 0 & m \end{vmatrix} = m^2 u = m[2m\epsilon - (\mu - eA)^2]^{1/2} .$$

Writing $F(\epsilon,\mu)$ for $f(x,u,v)$ when expressed as a function of the indicated variables we obtain

$$(3.8) \qquad n = \frac{2}{m} \int [2m\epsilon - (\mu - eA)^2]^{-1/2} F(\epsilon,\mu) d\epsilon d\mu$$

$$(3.9) \qquad J = \frac{2e}{m^2} \int (\mu - eA)[2m\epsilon - (\mu - eA)^2]^{-1/2} F(\epsilon,\mu) d\epsilon d\mu$$

$$(3.10) \qquad P_{xx} = \frac{2}{m^2} \int [2m\epsilon - (\mu - eA)^2]^{1/2} F(\epsilon,\mu) d\epsilon d\mu$$

The range of integration is in all cases given by (2.6). The dependence of n, J, and P_{xx} on A enters explicitly in the integrand and in the limits of integration. The integration domain is indicated by the dashed line in Fig. 5 (for $eA > 0$); it overlaps both free and trapped contributions.

It is important to note that a boundary condition on f at minus infinity restricts $F(\epsilon,\mu)$ only within the parabola $2m\epsilon > \mu^2$. For example, taking f to be isotropic at minus infinity implies that F is a function of ϵ alone, not everywhere, but only within this parabola. In particular, the condition that there exist no trapped particles ($F \equiv 0$ for $2m\epsilon < \mu^2$) precludes the possibility that F be isotropic anywhere but at minus infinity.

In (3.10), P_{xx} is expressed directly as a function of A. We now wish to compute dP_{xx}/dA, remarking that the domain of integration as well as the integrand depend on A. However, since the integrand happens to vanish on the boundary of the domain of integration, the only contribution comes from differentiation of the integrand,[6] viz.

$$(3.11)\qquad \frac{dP_{xx}}{dA} = \frac{2}{m^2}\int \frac{\partial}{\partial A}\,[2m\epsilon-(\mu-eA)^2]^{1/2}\;F(\epsilon,\mu)d\epsilon d\mu$$

$$= J$$

Thus we verify the equivalence of the two self-consistency formulations

$$(3.12)\qquad -\frac{1}{\mu_o}\frac{d^2A}{dx^2} = J(A)$$

$$(3.13)\qquad P_{xx}(A) + \frac{1}{2\mu_o}\left(\frac{dA}{dx}\right)^2 = P_o$$

The differential equation for the determination of $A(x)$ is exactly that of the motion of a particle in a given potential; we interpret x as time, A as position, $B = dA/dx$ as velocity, and $\mu_o P_{xx}(A)$ as the potential which is a given function of position. Thus we can determine the properties of the solution $A(x)$ from monotonicity and similar qualitative properties of the function $P_{xx}(A)$. What we shall do is discuss functions $P_{xx}(A)$ which yield satisfactory solutions $A(x)$, and then discuss the nature of the functions $F(\epsilon,\mu)$ which give rise to such $P_{xx}(A)$.

---

[6] Since we shall wish to consider discontinuous functions $F(\epsilon,\mu)$ and the integral is improper, this differentiation formula should be verified directly; this offers no difficulty.

First we remark that the constant P_o is the value taken by P_{xx} at minus infinity (i.e., at $A = 0$); if f happens to be isotropic at minum infinity, then P_o is the gas pressure. The condition $J = 0$ at minus infinity is a condition on F, viz.,

$$\int \mu[2m\epsilon - \mu^2]^{-1/2} F(\epsilon,\mu)d\epsilon d\mu = 0.$$

This is satisfied if, for example, F is an even function of μ (at minus infinity, f is an even function of v). More specially, this is satisfied if f is isotropic (F is independent of μ) at minus infinity. For simplicity, we shall henceforth consider F to be even in μ; this is no restriction on trapped orbits since they occur for only one sign of μ.

We have already determined that $P_{xx}(0) = P_o$ and $P'_{xx}(0) = J(0) = 0$; (the accent denotes a derivative with respect to A). We also wish to have $P_{xx}(\infty) = 0$; (there is no gas at plus infinity). The simplest possibility would be to have $P_{xx}(A)$ monotone (Fig. 7). In this case the solution to the differential equation (3.13) evidently has a monotone analogue "velocity", $B = dA/dx$, which increases from the value zero at $A = 0$ to the limiting value B_1 given

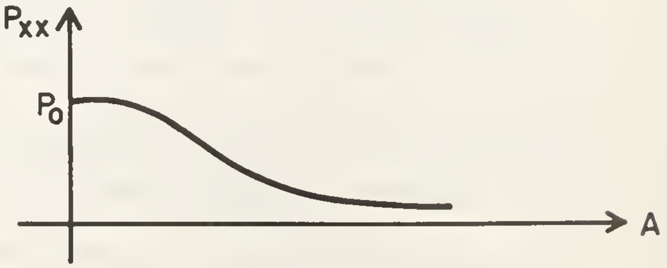


FIG. 7

by

$$B_1^2/2\mu_o = P_o \tag{3.14}$$

as A approaches infinity. Now consider x to decrease from plus infinity; i.e., the "ball rolls up the hill" in Fig. 7. As A decreases to zero, both $B = dA/dx$ and $J = -(d^2A/dx^2)/\mu_o$ approach zero. In the "normal" case that the curvature of the potential function is finite [$P''_{xx}(0)$ is finite], then just as for a pendulum, it takes an infinite amount of "time" to reach the summit; $A = 0$ is attained at $x = -\infty$. On the other hand, singular behavior in the form of an infinite curvature can yield the unusual result $A = dA/dx = d^2A/dx^2 = 0$ at a finite position, x_o. In this case, the complete solution to the problem consists in piecing together

$$\begin{cases} A \equiv 0, & x \leq x_o \\ A > 0, & x > x_o. \end{cases} \tag{3.15}$$

Since there is no magnetic field to the left of $x = x_o$, the particle orbits are rectilinear and all gas properties are constant. In other words, it is irrelevant whether we apply the boundary condition at $x = -\infty$ or at $x = x_o$ or at any intermediate point. To confirm that this analytically artificial function solves our problem, it is only necessary to note that the orbits computed in this proposed field are smooth (the acceleration is at least continuously differentiable) and do indeed yield the appropriate, self-consistent, current.

To complete the discussion it is necessary to find what conditions on F guarantee that $P_{xx}(A)$ is monotone and under what conditions $P''_{xx}(0)$ is finite or infinite. This is done by the following three theorems which are proved in Appendices I and II.

Theorem 1. If $F(\epsilon,\mu)$ is even in μ, and for each ϵ, F is a monotone decreasing function of $|\mu|$, then $P_{xx}(A)$ is monotone decreasing.

In the special case of an isotropic f at minus infinity, this requires that the number of trapped particles of a given energy decreases as we move to the right. More specially, P_{xx} is monotone for an isotropic distribution at infinity and no trapped particles.

Theorem 2. If $F(\epsilon,\mu)$ is a sufficiently smooth function (e.g., continuously differentiable), then J is of order A for small A and $P''_{xx}(0)$ is finite.

Theorem 3. If $F(\epsilon,\mu)$ is discontinuous across the curve separating trapped from free particles, then J is of order $\sqrt{A}$ for small A.

In particular, an isotropic distribution at infinity with no trapped particles is necessarily discontinuous.

4. Calculation of the "Thinnest" Profile

We now consider the special case in which there are no trapped particles,

$$(4.1) \qquad F(\epsilon,\mu) = 0, \quad \epsilon < \mu^2/2m.$$

The simplest way to incorporate this condition is to cut off the integration at $\epsilon = \mu^2/2m$. Combining this with the inequality (3.6), introducing the new variables α and σ (as in Appendix I),

$$(4.2) \qquad \begin{cases} \alpha^2 = 2m\epsilon \\ \sigma = \mu - eA \end{cases}$$

and taking $e > 0$ (there is no loss of generality), we obtain the range

$$(4.3) \qquad \begin{cases} -\alpha < \sigma < \alpha - eA \\ \alpha > \frac{1}{2}\, eA. \end{cases}$$

We have

$$
(4.4)\qquad \left\{
\begin{aligned}
n &= \frac{2}{m^2}\int_{\frac{1}{2}eA}^{\infty} \alpha d\alpha \left\{\int_{-\alpha}^{\alpha-eA} (\alpha^2-\sigma^2)^{-1/2}\,\emptyset(\alpha,\ \sigma+eA)d\sigma\right\} \\
J &= \frac{2e}{m^3}\int_{\frac{1}{2}eA}^{\infty} \alpha d\alpha \left\{\int_{-\alpha}^{\alpha-eA} (\alpha^2-\sigma^2)^{-1/2}\,\emptyset(\alpha,\ \sigma+eA)d\sigma\right\} \\
P_{xx} &= \int_{\frac{1}{2}eA}^{\infty} \frac{2}{m^3}\left\{\int_{-\alpha}^{\alpha-eA} (\alpha^2-\sigma^2)^{1/2}\ \emptyset(\alpha,\ \sigma+eA)d\sigma\right\},
\end{aligned}
\right.
$$

where we have set $\emptyset(\alpha,\mu)$ for $F(\alpha^2/2m,\ \mu)$.

For the special case of an isotropic distribution at minus infinity, $\emptyset$ is a function of α alone, and one integration can be performed explicitly,

$$
(4.5)\qquad P_{xx} = \frac{1}{m^3}\int_{\frac{1}{2}eA}^{\infty}\left\{(\alpha-eA)(2\alpha eA-e^2A^2)^{1/2} + \right.
$$

$$
\left. + \alpha^2\left[\frac{\pi}{2} + \sin^{-1}(1-eA/\alpha)\right]\right\}\emptyset(\alpha)\alpha d\alpha.
$$

In this case, the monotonicity with respect to A (cf. Appendix I) can be inferred directly from the monotonicity of the integrand.

From Appendix II, we take the behavior for small A,

$$
(4.6)\qquad J = -\varkappa A^{1/2} + O(A)
$$

where

$$(4.7) \qquad \varkappa = \frac{1}{m^3}\int_0^\infty (2e\alpha)^{3/2}\, \phi(\alpha,\alpha)d\alpha \quad ;$$

$-\phi(\alpha,\alpha)$ is the jump in $\phi(\alpha,\mu)$ as described in the Appendix. For an isotropic distribution, $\phi(\alpha,\alpha)$ is merely $\phi(\alpha)$. From this we conclude,

$$(4.8) \qquad P_{xx} - P_o = -\frac{2}{3}\varkappa A^{3/2} + O(A^2).$$

The solution of the differential equation (3.13) in the neighborhood of $A = 0$, viz

$$\frac{1}{2\mu_o}\left(\frac{dA}{dx}\right)^2 = \frac{2}{3}\varkappa A^{3/2},$$

is then

$$(4.9) \qquad \begin{cases} A = \frac{1}{36}\mu_o^2\varkappa^2(x-x_o)^4 + \ldots \\ B = \frac{1}{9}\mu_o^2\varkappa^2(x-x_o)^3 + \ldots \\ J = -\frac{1}{3}\mu_o\varkappa^2(x-x_o)^2 + \ldots \end{cases}$$

We see that not only the current but even its derivative is continuous at the join.

Possibly the most interesting special case is that of an isotropic Maxwellian distribution at minus infinity,

$$(4.10) \qquad \left\{ \begin{array}{l} F(\epsilon) = \dfrac{mn_o}{2\pi kT_o} \exp(-\epsilon/kT_o) \\[2ex] \emptyset(\alpha) = \dfrac{mn_o}{2\pi kT_o} \exp(-\alpha^2/2mkT_o). \end{array} \right.$$

The expression for P_{xx} is not appreciably simpler except that the scaling can be shown explicitly,

$$(4.11) \qquad P_{xx}/p_o = \mathcal{P}(s) \equiv \frac{s^4}{2\pi} \int_{1/2}^{\infty} Q(\beta) \exp(-\tfrac{1}{2}\beta^2 s^2)\beta d\beta$$

where

$$(4.12) \qquad Q(\beta) = (\beta-1)(2\beta-1)^{1/2} + \beta^2[\tfrac{\pi}{2} + \sin^{-1}(1 - 1/\beta)]$$

and

$$(4.13) \qquad s = eA/(mkT_o)^{1/2}.$$

The dimensionless vector potential s can be interpreted as distance measured in local Larmor radii,

$$(4.14) \qquad \frac{ds}{dx} = eB/(mkT_o)^{1/2}$$

The differential equation for s takes the form

$$(4.15) \qquad \frac{1}{2}\left(\frac{ds}{d\xi}\right)^2 = 1 - \mathcal{P}(s)$$

where

$$(4.16)\qquad \begin{cases} \xi = x/\lambda \\ \lambda^2 = \dfrac{m}{\mu_o n_o e^2} \;. \end{cases}$$

Except for scaling, there is essentially one magnetic profile to be computed. The scale length, λ, can be interpreted variously as the ratio of the speed of light to the appropriate (ion or electron) plasma frequency, or as a Larmor radius composed of the magnetic field strength at plus infinity and the mean thermal speed at minus infinity.

The constant $\varkappa$, can be evaluated explicitly (cf. (4.7) and (4.10)), and we find that near x_o,

$$(4.17)\qquad B/B_1 = \frac{2}{9\pi^2}\,\Gamma^2(5/4)(\xi-\xi_o)^3 \sim [0.123(\xi-\xi_o)]^3 \;;$$

the initial scale length is about eight times λ.

Certain singular distributions exhibit interesting mathematical features. For example, a two-dimensional constant speed distribution,

$$(4.18)\qquad F(\epsilon) = \frac{mn_o}{2\pi}\,\delta(\epsilon-\epsilon_o),$$

allows P_{xx} to be computed explicitly,

$$(4.19)\qquad \begin{cases} P_{xx} = 0 \;, \qquad \frac{1}{2}\,eA > \alpha_o \\ P_{xx} = \frac{1}{\pi}\, n_o\epsilon_o \Big\{ (1-eA/\alpha_o)(2eA/\alpha_o - e^2A^2/\alpha_o^2)^{1/2} \\ \qquad + \frac{\pi}{2} + \sin^{1-}(1 - eA/\alpha_o) \Big\} \;, \qquad \frac{1}{2}\,eA < \alpha_o \;. \end{cases}$$

It is worth remarking that any truncated distribution function, i.e., one which is zero for sufficiently large energy, exhibits this feature, that P_{xx} vanishes for sufficiently large A. Such a distribution will therefore yield a finite transition; both plasma and field are together in only a finite layer.

A more interesting singular distribution is one which is singular in μ,

$$(4.20) \qquad \phi(\alpha,\mu) = \psi(\alpha)\delta(\mu).$$

This contains particles all of which are initially aimed perpendicular to the magnetic field.

An elementary computation yields the exact results

$$(4.21) \qquad \left\{ \begin{aligned} n &= \frac{2}{m^2}\int_{eA}^{\infty} (\alpha^2 - e^2A^2)^{-1/2}\psi(\alpha)\alpha d\alpha \\ J &= -\frac{2e^2A}{m^3}\int_{eA}^{\infty} (\alpha^2 - e^2A^2)^{-1/2}\psi(\alpha)\alpha d\alpha = -\frac{e^2A}{m}n, \end{aligned} \right.$$

and the expansion

$$(4.22) \qquad P_{xx} = p_o - \frac{n_o e^2}{2m}A^2 + \dots \ .$$

Thus, for this very singular velocity distribution, P_{xx} is regular near $A = 0$ and the magnetic field extends to minus infinity, decaying exponentially. If $\psi(\alpha)$ is the one-dimensional Maxwellian distribution,

$$\psi = n_o m \left(\frac{m}{2\pi k T_o}\right)^{1/2} \exp(-\alpha^2/2mkT_o) \tag{4.23}$$

then we have, explicitly,

$$\left\{\begin{aligned} n &= n_o \exp(-e^2A^2/2mkT_o) \\ J &= -(e^2 n_o A/m) \exp(-e^2A^2 2mkT_o) \\ P_{xx} &= p_o \exp(-e^2A^2/2mkT_o). \end{aligned}\right. \tag{4.24}$$

This result for a distribution which is singular in the y-component of velocity, v, is extremely special. If, instead of a concentrated distribution at $v = 0$, we take a distribution concentrated at $v = v_o \neq 0$ (to maintain $J = 0$ at infinity we take the sum of two such concentrated distributions at $v = \pm v_o$), then we return to the general result, $J \sim A^{1/2}$, and the magnetic field no longer penetrates to minus infinity.

Appendix I: Monotonicity of $P_{xx}(A)$

We assume that $F(\epsilon,\mu)$ is an even function of μ and is a non-increasing function of $|\mu|$. We wish to show that J is negative for $A > 0$. This, of course, implies that P_{xx} is monotone. The expression (3.9) for J can be written

$$(I.1) \qquad J = \frac{2e}{m^3}\int_0^\infty \alpha d\alpha \left\{ \int_{eA-\alpha}^{eA+\alpha} (\mu-eA)[\alpha^2-(\mu-eA)^2]^{-1/2} \phi(\alpha,\mu)d\mu \right\}$$

where we have introduced the variable α instead of ϵ,

$$(I.2) \qquad \alpha^2 = 2m\epsilon ,$$

$\phi(\alpha,\mu)$ for $F(\alpha^2/2m,\mu)$, and the explicit range of integration (3.6). Setting

$$(I.3) \qquad \sigma = \mu - eA ,$$

the inner integral becomes

$$\int_{-\alpha}^{\alpha} \sigma(\alpha^2 - \sigma^2)^{-1/2} \phi(\alpha,eA+\sigma)d\sigma$$
$$= \int_0^{\alpha} \sigma(\alpha^2 - \sigma^2)^{-1/2} [\phi(\alpha,eA+\sigma)-\phi(\alpha,eA-\sigma)]d\sigma .$$

The bracket in this last expression has a sign opposite to that of eA. For $eA > 0$, $|eA+\sigma| \geq |eA-\sigma|$, thus $\phi(\alpha,eA+\sigma) \leq \phi(\alpha,eA-\sigma)$, and conversely for $eA < 0$. Noting that there is a factor e in the expression for J, we conclude that J is non-positive. This establishes that P_{xx} is monotone decreasing in the weak sense. For a given value of $A > 0$, the inequality will hold, $J < 0$, unless $\phi(\alpha, eA+\sigma) = \phi(\alpha, eA-\sigma)$ for all $\alpha > 0$ and all σ, $0 < \sigma < \alpha$. In other words, $J(A) = 0$ only if ϕ is a function of α alone in the range $eA-\alpha < \mu < eA+\alpha$, i.e., f is isotropic for all particles, free or trapped, whose orbits are accessible to the given value of A. If the distribution of free particles is non-isotropic, then $J < 0$ for every $A > 0$ and there is strict monotonicity of P_{xx}. Or, if the distribution of free particles is isotropic but there is a strict inequality between the number of trapped and free particles for any value of the energy, again there is strict monotonicity of P_{xx}.

Appendix II: Behavior of $P_{xx}(A)$ for Small A

Introducing the same variables as in Appendix I, we have

$$\text{(II.1)} \qquad J(A) = \frac{2e}{m^3}\int_0^\infty \alpha d\alpha \left\{\int_{-\alpha}^{\alpha} \sigma(\alpha^2 - \sigma^2)^{-1/2} \phi(\alpha, eA+\sigma) d\sigma\right\}.$$

If ϕ is a sufficiently regular function of its second argument then so is $J(A)$; successive derivatives can be obtained by differentiation under the integral sign. In particular, if ϕ is differentiable with respect to μ then the derivative $J'(0) = P''_{xx}(0)$ exists.

Now let us consider the case of a function $\phi(\epsilon,\mu)$ which is discontinuous, specifically across the parabola $\epsilon = \mu^2/2m$ which separates free from trapped particles. For simplicity we write

$$\phi = \phi_0 + \phi_1$$

where ϕ_0 is regular over some large domain and ϕ_1 vanishes on the free orbits, $\epsilon > \mu^2/2m$. We have two contributions to J.

$$J = J_0 + J_1,$$

where, as before, J_0 is at least of order A. For J_1 we evaluate only the trapped orbits (for simplicity taking the case $e > 0$),

$$J_1 = \frac{2e}{m^3} \int_0^\infty \alpha d\alpha \left\{ \int_{\alpha - eA}^{\alpha} \sigma(\alpha^2 - \sigma^2)^{-1/2} \phi_1(\alpha, eA+\sigma) d\sigma \right\}.$$

For continuous ϕ_1, the inner integral can be approximated near $A = 0$ by

$$\phi_1(\alpha,\alpha) \int_{\alpha-eA}^{\alpha} \sigma(\alpha^2 - \sigma^2)^{-1/2} d\sigma = \phi_1(\alpha,\alpha)(2\alpha eA - e^2A^2)^{1/2}$$

$$\sim \phi_1(\alpha,\alpha)(2\alpha eA)^{1/2}.$$

This gives the dominant contribution to J, of order $A^{1/2}$,

$$J = \frac{(2e)^{3/2}}{m^3} A^{1/2} \int_0^\infty \phi_1(\alpha,\alpha)\alpha^{3/2} d\alpha + O(A).$$

The quantity $\phi_1(\alpha,\alpha)$ is merely the jump in $\phi(\alpha,\mu)$ on the parabola $\mu = (2m\epsilon)^{1/2}$, or straight line $\mu = \alpha$.

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